THE..

SCATTERING

..of..

X--Rays.

by

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SCATTERING OF X-RAYS.

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The Scattering of X-Rays.

Introduction and brief review of some previous work.

1. One of the outstanding problems in electromagnetic radiation which thus far has not received a satisfactory solution is the one offered by the phenomenon of the so-called "scattered X-rays". Many attempts have been made, on the basis of classical electrodynamics, to derive an expression giving the correct amount of energy scattered by various elements under various conditions(e.g., for different wave-lengths). However, it cannot be said that any of these attempts has met with success.

2. The first attempt to give a theoretical explanation of the scattering of X-rays was that of Sir J.J.Thomson. On assuming that

1. Classical electromagnetic theory is applicable to the problem;
2. Each electron scatters independently,
3. There are no other forces acting on the electrons which are comparable in magnitude with the forces due to the incident beam,

4. The dimensions of the electron~~s~~ are negligible compared with the wave-length of the incident radiation,

Professor Thomson[1] showed that the mass scattering coefficient of any substance is given by the expression

$$\sigma = \frac{8\pi e^4 N p}{3 m^2 c^4}, \qquad (1)$$

where N is the number of atoms per cc., p the number of electrons per atom, e the electronic charge, m the mass of an electron, and c the velocity of light. The amount of energy received per second per unit area is defined as the intensity of radiation. According to Thomson's theory the intensity of the beam scattered by an electron at an angle θ with the incident beam is given by

$$I_\theta = I \frac{e^4(1+\cos^2\theta)}{2 r^2 m^2 c^4}, \qquad (2)$$

where I is the intensity of the incident beam, and r the distance between the center of the electron and the point at which we are calculating the intensity of the scattered beam (see FIG.1).

3. From formulas (1) and (2) the conclusions are drawn that (a) the coefficient of scattering should be independent of the wave-length, (b) the mass scattering coefficient should not in any case be less than .20

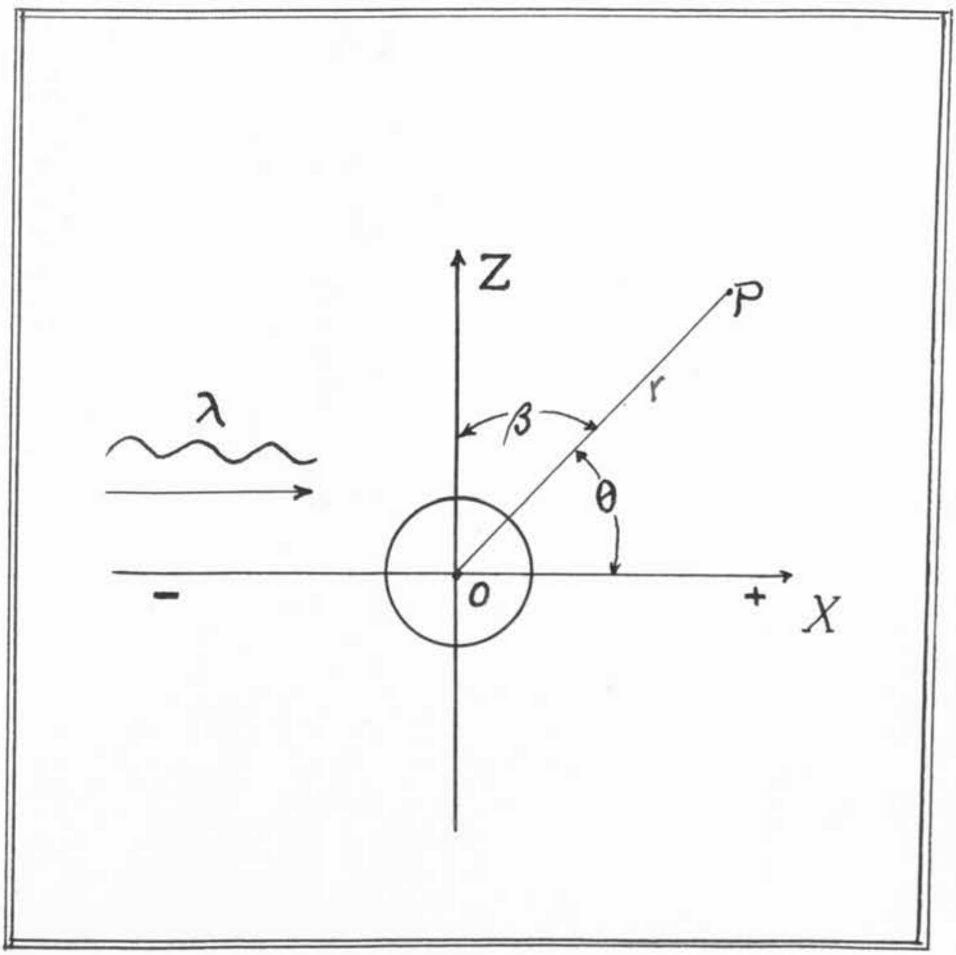


FIG. 1.

if the number of electrons in atoms other than those of Hydrogen is equal to the atomic number, and (c)the dis-tribution of the intensity of the scattered beam should be symmetrical with respect to the scattering plate(ra-diator).

4. The experiments of Barkla[2] and Dunlop show that for a considerable range in wave-lengths of X-rays the mass scattering coefficients of elements of low atomic weight are correctly given by (1) if the number of electrons is taken to be equal to the atomic number. This assumption has been shown[3] to be valid for the lighter elements when X-rays of ordinary hardness are used. But, for elements of high atomic weight and soft X-rays, the total scattering is much greater than would be ex-pected from Thomson's formula. On the other hand, Barkla[4] and White have shown that for wave-lengths less than 2×10^{-9}cm. the total mass absorption coefficient of a light element is less than the theoretical value of the mass scattering coefficient alone. Moreover, Ishino[5] has shown that in the case of radiation of high frequency the value of the mass scattering coefficients for Al, Fe, and Pb are only one fourth of the values calculated on the basis of Thomosn's theory.

5. The fact that, when soft X-rays are used, elements of high atomic weight scatter to a greater extent than

would be predicted by the classical theory has been accounted for by assuming[6] that the electrons do not scatter independently when the wave-length of the incident radiation is comparable with the dimensions of the atom. But classical electrodynamics has not been able to account for the diminution in the scattering when hard X-rays are used.

6. It has already been noticed that according to Thomson's theory the intensity of the scattered radiation should be symmetrically distributed with respect to the radiator as shown in FIG.2. For those elements of low atomic weight whose mass scattering coefficients are given correctly by (1), Barkla[7] has shown that for a certain range in wave-lengths this prediction is fulfilled. But, for relatively soft X-rays and hard γ- rays this prediction has been shown[8] not to be valid.

7. When heavy elements and incident radiation of long wave-lengths are used, the dissymmetry is accompanied by an increase in the total energy scattered. This "excess scattering" is usually explained by assuming that in heavy elements the electrons are "closely packed", so that the rays scattered by the individual electrons are almost in the same phase.

8. However, this explanation cannot be applied to the

Theoretical Distribution
of the Intensity of Scattered X-rays
according to Thomson's Theory.

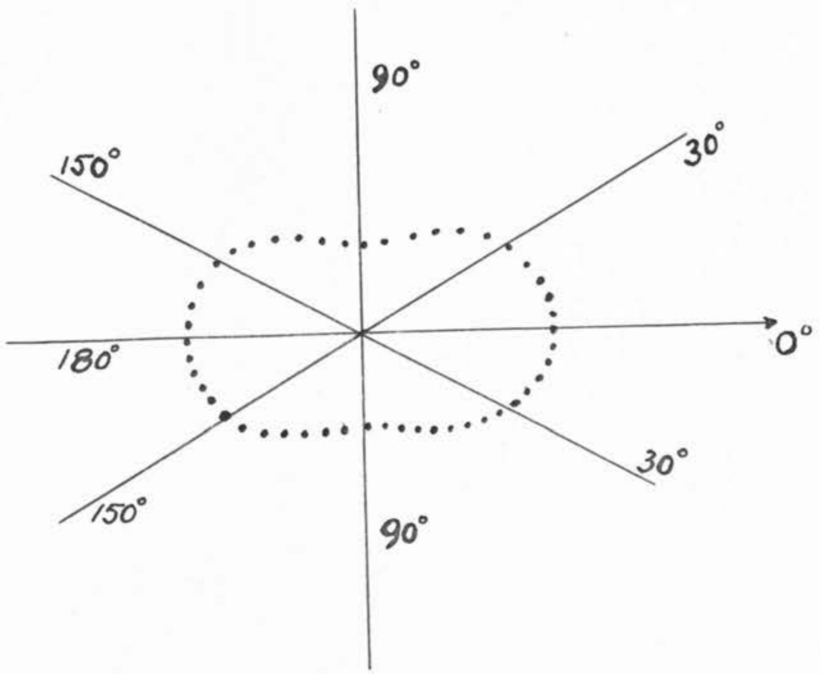


FIG. 2.

asymmetrical scattering of hard γ-rays.[9] For the asymmetry in this case is accompanied by a diminution in the total energy scattered and not an increase as we should expect. As Compton[9] points out, one is thus led to the conclusion that the asymmetrical scattering of short waves is due to some property of the electron itself.

9. The general problem of the scattering of X-rays by groups of electrons has been studied by Debye[10], Schott[11] Glocker and Kaupp[12], and A.H.Compton[13]. In their essential features the theories of these investigators do not differ greatly. We shall therefore discuss briefly only the theory of G.A.Schott and rather more fully that of Debye as we shall make use of the latter theory in the present work.

SCHOTT'S THEORY.

10. Schott assumes that the atom consists of coaxal electron rings. The electrons in each ring are at equal distances from each other, and revolve with a uniform angular velocity which, however, may be different for different rings. In this theory, as well as in those of Thomson and Debye, it is assumed that classical electrodynamics is applicable. The effect of the magnetic forces in the incident beam, the reaction due to the radiation

from the electrons, the variation of mass with speed, and the mutual interaction of the different rings are neglected . The assumption that the electron rings scatter independently makes it unnecessary to take into account the multiple scattering which undoubtedly occurs to some extent, and which would have the effect of diffusing the radiation and so decrease the asymmetry.

11. The final results arrived at are that (1)in the case of Hydrogen the predictions concerning the scattering coefficient and asymmetry are the same as those of the classical theory, (2)for atoms containing rings of more than one electron there should be fore and aft asymmetry, and the scattering coefficient should increase with increasing wave-length up to a vlaue not much less than p times that given by the classical theory, where p is the number of electrons in the ring; (3)the character of the scattering and asymmetry depends on the number of rings in the atom and the number of electrons in each ring; (4)the minimum value of the scattering coefficient on this theory is that given by the classical theory; and (5)this theory is altogether incapable of explaining the exceptionally small scattering coefficients observed by Ishino[5].

DEBYE'S THEORY.

12. Debye's theory, in its essentials, has the same merits and demerits as that of Schott. It also accounts for the dissymmetry and the excess scattering, but is totally unable to account for the saml1 scattering coefficients observed by Ishino, Barkla, and others.

13. Debye assumes that all the electrons in an atom are arranged in a single ring of radius a, and that they are equally distant from each other. He further assumes that the only effective forces are those of the incident beam.

14. Let the incident beam, which is assumed to be polarized in the Z-direction, be directed along the X-axis of a right-handed co-ordinate system. The components of the electric vector as given by Debye[10] are:

$$\left.\begin{aligned} E_x &= -\alpha\gamma\frac{e^2}{mc^2}E_0\frac{\varepsilon^{i(\omega t-kR)}}{R}\cdot\varepsilon^{ik\eta_n}, \\ E_y &= -\beta\gamma\frac{e^2}{mc^2}E_0\frac{\varepsilon^{i(\omega t-kR)}}{R}\cdot\varepsilon^{ik\eta_n}, \\ E_z &= (1-\gamma^2)\frac{e^2}{mc^2}E_0\frac{\varepsilon^{i(\omega t-kR)}}{R}\cdot\varepsilon^{ik\eta_n}, \end{aligned}\right\} \quad (3)$$

where

$$\eta_n \equiv [(\alpha-1)x_n+\beta y_n+\gamma z_n],$$

$$k \equiv \omega/c,$$

R is the distance from the center of the ring to the point at which the intensity is being calculated,

α, β, γ, are the direction cosines of the vector joining this point and the oringin, e the electronic charge, ε the Naperian base, and x_n, y_n, z_n the co-ordinates of the nth electron in its position preceding the disturbance by the incident beam. For the whole field and at great distances we will have[10]

$$\left.\begin{aligned} E_x &= -\alpha\gamma \frac{e^2}{mc^2} E_0 \frac{\varepsilon^{i(\omega t - kR)}}{R} \sum \varepsilon^{ik\eta_n}, \\ E_y &= -\beta\gamma \frac{e^2}{mc^2} E_0 \frac{\varepsilon^{i(\omega t - kR)}}{R} \sum \varepsilon^{ik\eta_n}, \\ E_z &= (1-\gamma^2) \frac{e^2}{mc^2} E_0 \frac{\varepsilon^{i(\omega t - kR)}}{R} \sum \varepsilon^{ik\eta_n}. \end{aligned}\right\} \quad (4)$$

The energy is proportional to E^2 where

$$E^2 = E_x^2 + E_y^2 + E_z^2 \quad . \qquad (5).$$

Debye[10] has shown that, if subscripts n and m refer to the nth and mth electrons, the ratio of the intensity of the radiation scattered by one atom at a large distance R, to that of the energy contained per sq.cm. in the incident beam is given by

$$v = \frac{(1+\alpha^2)e^4}{2R^2m^2c^4} \sum\sum \varepsilon^{ik[(\alpha-1)(x_n-x_m)+\beta(y_n-y_m)+\gamma(z_n-z_m)]}. \quad (6)$$

15. In order to find the scattering by many molecules with electron rings we note first that for

an isomorphous body, all orientations of the atoms (molecules) are equally likely. Thus, if we consider a portion of the radiator, say one cc., in which there are N atoms, the intensity of the scattered radiation in a direction whose direction-cosines are α, β, γ may be found by first finding the mean value of v for one atom for all possible orientations and then multiplying the result by N. The final result obtained by Debye is that the ratio of the **intensity** of the energy scattered by N atoms to that of the incident beam is given by

$$V = \frac{Npe^4(1+\alpha^2)}{2R^2m^2c^4}\sum_{n=0}^{p-1}\frac{\sin[4ka\sin\frac{n\pi}{p}\sin\frac{\theta}{2}]}{[4ka\sin\frac{n\pi}{p}\sin\frac{\theta}{2}]}, \qquad (7)$$

where θ is the angle between the incident beam and the direction at which V is calculated, N the number of atoms per cc., and p the number of electrons per atom.

16. Let λ be the wave-length of the incident radiation Then, from the definition of k, we will have

$$ka = \frac{2\pi a}{\lambda} \qquad (8)$$

The following special cases are of interest:

Case 1: If λ is large in comparison with the radius of the electon ring(which, in the case of Hydrogen is about $.72 \times 10^{-8}$cm.)so that $ka \ll 1$, then each term under the summation sign in (7) approaches 1, and

$$V = \frac{(1+\cos^2\theta)e^4Np^2}{2R^2m^2c^4} \qquad \ldots\ldots\ldots (9)$$

Case 2: On the other hand, if $ka >> 1$; i.e., if $\lambda << a$ and θ is sufficiently large,

$$V = \frac{(1+\cos^2\theta)e^4Np}{2R^2m^2c^4} \qquad \ldots\ldots\ldots (10)$$

17. The coefficient of scattering, σ, for the two special cases may be found by integration. The results are:

Case 1: $\lambda >> a$.

$$\sigma = \frac{8\pi Np^2e^4}{3m^2c^4} \qquad \ldots\ldots\ldots (11)$$

Case 2: $\lambda << a$.

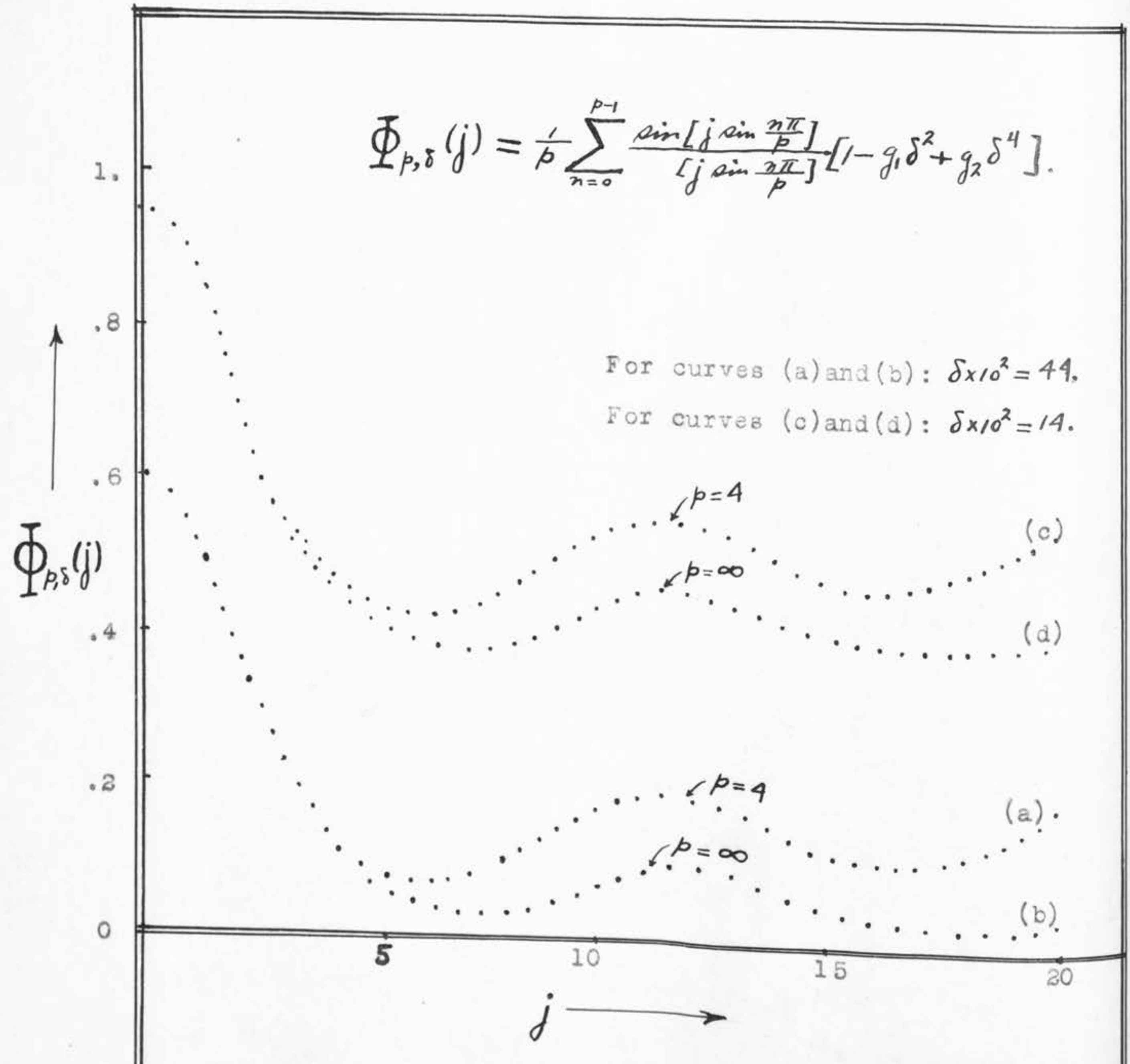
$$\sigma = \frac{8\pi Npe^4}{3m^2c^4} \qquad \ldots\ldots\ldots (12)$$

In the general case σ will be given by

$$\sigma = \int VR^2 d\Omega. \qquad \ldots\ldots\ldots (13)$$

where $d\Omega$ is an element of solid angle.

18. The minimum amount of scattering according to Debye's theory is far in excess of the observed values. However, as has been mentioned, the theory furnishes a plausible explanation of the fore and aft asymmetry and the accompanying excess radiation. The dissymmetry will depend on the numbe of electrons in the ring. The function



$\Phi_{p,\delta}(j) = \frac{1}{p}\sum_{n=0}^{p-1}\frac{\sin[j\sin\frac{n\pi}{p}]}{[j\sin\frac{n\pi}{p}]}[1-g_1\delta^2+g_2\delta^4]$.
For curves (a)and(b): $\delta\times10^2 = 44$.
For curves (c)and(d): $\delta\times10^2 = 14$.
$p=4$
$p=\infty$
(c)
(d)
$p=4$
(a)
$p=\infty$
(b)
1.
.8
.6
.4
.2
0
5
10
15
20
$\Phi_{p,\delta}(j)$
j

FIG. 3.

$$\psi_p(j) = \frac{1}{p}\sum_0^{p-1} \frac{\sin[j \sin\frac{n\pi}{p}]}{[j \sin\frac{n\pi}{p}]},$$

where $j = 4ka \sin\frac{\theta}{2}$, when plotted against j would have the same form as the curves given in FIG.3 except that each ordinate would be magnified slightly and by an equal amount. FIG.4 shows the theoretical distribution of intensity according to Debye's and Thomson's theories. In FIG.5 Crowther's[14] experimentally determined curve for the distribution of intensity round an Al radiator is compared with the theoretical distribution according to Debye's theory when Hydrogen is used as the scattering substance.

19. We have seen that experiments on the scattering of short waves lead us to the conclusion that the explanation of the small scattering coefficients is to be found in some property of the electron itself. An interesting attempt to do this is that of A.H.Compton[9]. Compton assumes that the radius of the electron is of the same order of magnitude as the wave-length of short γ-rays, and that the incident electromagnetic wave is capable of moving the different parts of the electron relatively to each other. He further postulates the validity of assumptions (1) to (3) given on page 1. The size of the electron being comparable

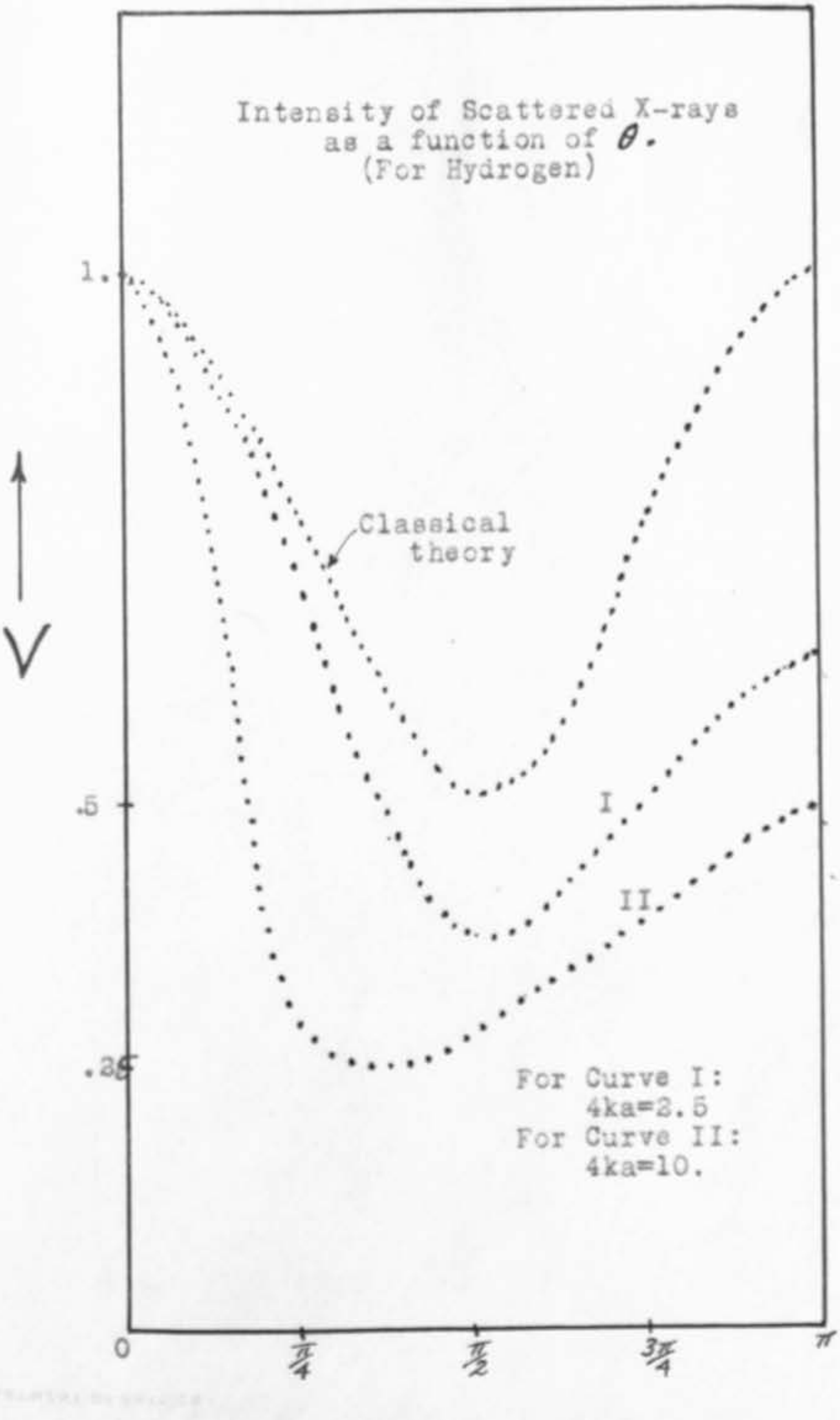
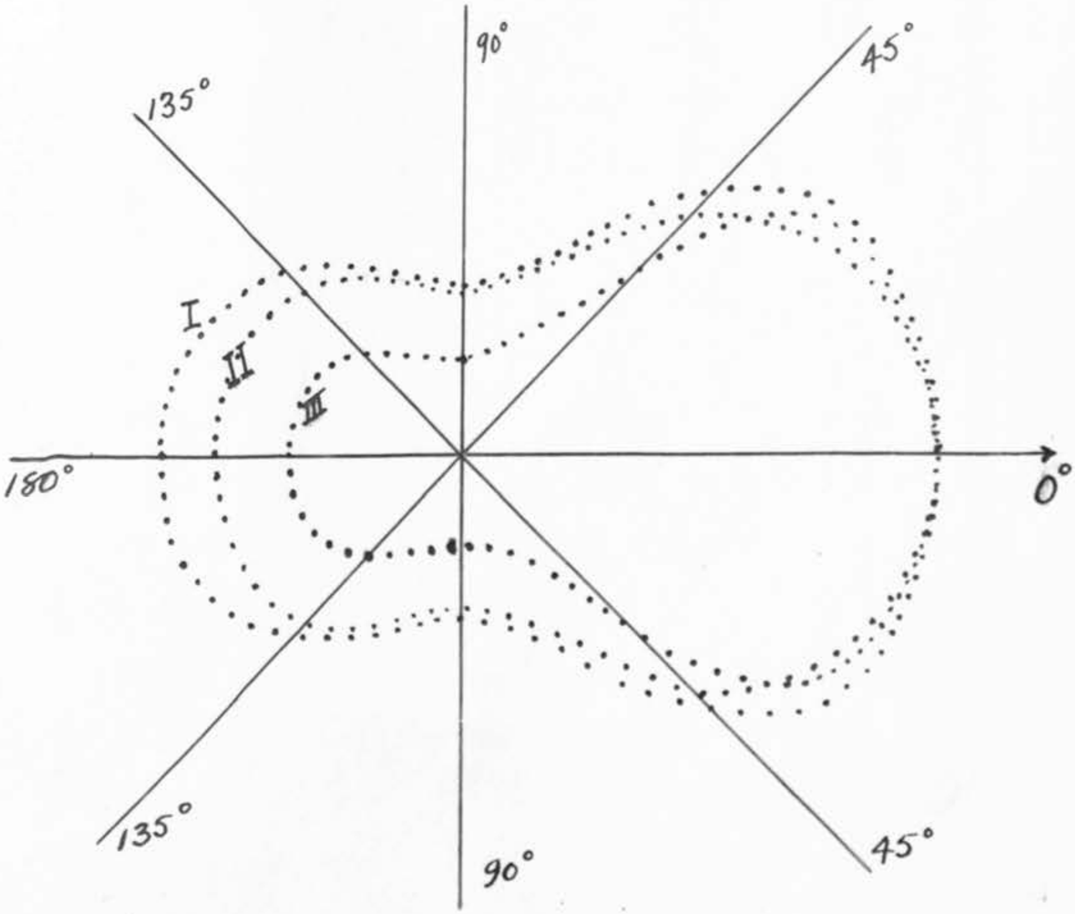

Intensity of Scattered X-rays
as a function of θ.
(For Hydrogen)
Classical
theory
I
II.
For Curve I:
4ka=2.5
For Curve II:
4ka=10.
1.
.5
0
π/4
π/2
3π/4
π
V
θ

FIG. 4.

with the wave-length used, the wave scattered by different parts of the electron will not all be in exactly the same phase. The introduction of the phenomenon of interference is found to lead to a decrease in the scattering coefficient and also to a fore and aft asymmetry in the distribution of the intensity of the scattered radiation.

20. Among the objections one might raise against the assumption of a large electron are that (1)recent experiments by Richtmyer[15] have given values of the mass absorption coefficient for Al(for different wave-lengths) which are from 7 to 90 per cent higher than would be expected on the basis of Compton's theory; (2)if we take the radius of Compton's electron as 2×10^{-10} cm. and use Millikan's value of e we find the mass of the electron to be only 1/1000 that obtained from the values of e and e/m; and (3)in assuming that each element of the electron has a definite mass independent of the rest of the electron and that there is relative motion among the parts of the electron, Professor Compton cannot call upon classical electrodynamics to guide him in finding the motion of his electron. For, classical electrodynamics gives us no information as to how an element of an electron moves. As regards the motion of the electron

Distribution of the Intensity
of Scattered X-rays.



I: Theoretical Curve for Hydrogen according to Debye's theory. (4ka=2.5)

II: Theoretical Curve for Hydrogen according to the modified Debye's theory. (4ka=2.5)

III: Crowther's experimental curve for Al.

FIG. 5.

as a whole the usual assumption is that of Lorentz; namely, that the electron moves in such a way that the external force acting on it is equal and opposite to that which the electron exerts on itself.

21. In the present reign of the Quantum theory one would naturally turn to it with the object of deriving whatever benediction it has so far as the scattering of X-rays is concerned. Without invoking hypotheses founded on the Quantum theory, however, it would be of interest to see whether the classical theory could be modified in such a way as to be able to explain the outstanding facts of the scattering of X-rays, in particular, the observed diminution in the scattering coefficient. The object of the present work is to present a discussion of the possibility of doing this.

22. We shall assume that the electron consists of a large number of parts----for analytical simplicity, of two parts----having equal charges but different masses. In addition, as regards the electron as a whole, we make the assumptions stated above when reviewing J.J. Thomson's theory.

23. We recall that the equation of motion of an electron is usually written as

(mass)(acceleration) = external force.

We shall modify this equation for each part of the electron by adding to the left-hand side a term depending on the relative displacements, and another term depending on the relative velocities of these parts. The introduction of the term depending on the relative velocities corresponds to introducing, in ordinary dynamics, a frictional term. Our usual ideas of friction, however, are altogether inapplicable when we consider parts of an electron. Nevertheless, this term plays the same rôle as the dissipativity function in dynamics and gives rise to some difficulties in connection with the energy equation. A brief discussion of these difficulties will be given below: for the present we shall give the workable features of the theory in some detail.

24. Let the incident beam, assumed to be polarized in the Z-direction, be directed along the X-axis of a right-handed system of co-ordinates(FIG.1). Let m_1, m_2, ξ_1, ξ_2, $(e/2)$, $(e/2)$ be the masses, displacements, and charges repectively of the two parts of the electron. The equations of motion of these parts are assumed to be of the form:

$$m_1 \ddot{\xi}_1 + b(\dot{\xi}_1 - \dot{\xi}_2) + a(\xi_1 - \xi_2) = \frac{eE_0}{2} \varepsilon^{i(\omega t - kx)}, \quad (14)$$

$$m_2 \ddot{\xi}_2 + b(\dot{\xi}_2 - \dot{\xi}_1) + a(\xi_2 - \xi_1) = \frac{eE_0}{2} \varepsilon^{i(\omega t - kx)}, \quad (15)$$

where $k = 2\pi\nu/c$, ν is the frequency of the incident radiation, c the velocity of light, a,b constants, and the other symbols have their usual significance. To solve these equations, assume

$$\xi_1 = A\varepsilon^{i(\omega t - kx)}, \quad \ldots\ldots\ldots (16)$$

$$\xi_2 = B\varepsilon^{i(\omega t - kx)}, \quad \ldots\ldots\ldots (17)$$

where A and B are to be determined. According to Classical theory the electric intensity at any point P distant r from the center of the electron and at an angle β between the vector OP and the electric vector of the incident beam is given by

$$X_P = \frac{e \sin\beta}{2rc^2}(\ddot{\xi}_1 + \ddot{\xi}_2). \quad \ldots\ldots (18)$$

The amplitude at the point P is given by the expression

$$J = -\frac{e \sin\beta}{2rc^2}\omega^2(A+B) = g\omega^2(A+B), \quad \ldots (19)$$

where, for brevity, we have put

$$g = -e\sin\beta/2rc^2.$$

From equations (14),(15),(16), and (17) we get the following equations:

$$-m_1 A\omega^2 + b(iA\omega - iB\omega) + a(A-B) = \frac{eE_0}{2}; \quad \cdots\cdots (21)$$

$$-m_2 B\omega^2 + b(iB\omega - iA\omega) + a(B-A) = \frac{eE_0}{2}; \quad \cdots\cdots (22)$$

i.e.,

$$A+B = \frac{-eE_0}{m_2\omega^2} + (1-\mu)A, \quad \cdots\cdots\cdots (23)$$

where

$$\mu \equiv \frac{m_1}{m_2}.$$

Solving equations (21),(22), and (23) for (A+B) we find, after a simple calculation, that

$$J = q\omega^2(A+B)$$

$$= \frac{q\omega^2 eE_0\left\{\left[a(1+\mu^2+6\mu) - (1+\mu)\left(m_1\omega^2 + \frac{4a^2}{m_2\omega^2} + \frac{4b^2}{m_2}\right)\right]^2 + b^2\omega^2(1-\mu)^4\right\}^{1/2}}{2\left[a^2(1+\mu)^2 - 2am_1\omega^2(1+\mu) + m_1\omega^4 + b^2\omega^2(1+\mu)^2\right]}$$

$$= \frac{-e^2\omega^2 E_0 \sin\beta}{4rc^2} Q, \quad \cdots\cdots (24)$$

where

$$Q = \frac{\left\{\left[a(1+\mu^2+6\mu) - (1+\mu)\left(m_1\omega^2 + \frac{4a^2}{m_2\omega^2} + \frac{4b^2}{m_2}\right)\right]^2 + b^2\omega^2(1-\mu)^4\right\}^{1/2}}{a^2(1+\mu)^2 - 2am_1\omega^2(1+\mu) + m_1\omega^4 + b^2\omega^2(1+\mu)^2} \quad \cdots\cdots (25)$$

25. The intensity of the radiation at the point P will be proportional to J^2; and the ratio of this intensity to that of the incident beam will be given by

$$v = \frac{e^4\omega^4 \sin^2\beta}{16r^2c^4} Q^2. \quad \cdots\cdots (26).$$

If the incident beam is unpolarized the electric vector will be uniformly distributed in the plane of the wave, so that the mean value of $\sin^2\beta$ should be used in calculating v. This is easily shown to be equal to $(1+\cos^2\theta)/2$, where θ is the angle between the incident beam and the vector joining the center of the electron and the point at which we are calculating v. Hence, for unpolarized light,

$$v = \frac{(1+\cos^2\theta)\,e^4\omega^4}{32r^2c^4}Q^2 \quad \text{. . . . . (27)}$$

The scattering coefficient per electron, σ_1, is found by integrating the last expression over a sphere of radius r. The result is

$$\sigma_1 = \frac{8\pi e^4}{3c^4}\left(\frac{\omega^4Q^2}{16}\right), \quad \text{- - - - - - - - (28)}$$

where Q depends only on the constants a, b, and the ratio, μ, of the masses m_1 and m_2.

26. To find the value of Q we proceed as follows. Let there be no external force. The equations of motion of the two parts of the electron will then be:

$$m_1\ddot{\xi}_1 + b(\dot{\xi}_1-\dot{\xi}_2) + a(\xi_1-\xi_2) = 0, \quad \text{. . . . . . (29)}$$

$$m_2\ddot{\xi}_2 + b(\dot{\xi}_2-\dot{\xi}_1) + a(\xi_2-\xi_1) = 0. \quad \text{. . . . . (30)}$$

Dividing equation (29) by m_1 and equation (30) by m_2, and subtracting (30) from (29) there results:

$$\frac{d^2}{dt^2}(\xi_1 - \xi_2) + b\left(\frac{m_1+m_2}{m_1m_2}\right)\frac{d}{dt}(\xi_1-\xi_2) + a\left(\frac{m_1+m_2}{m_1m_2}\right)(\xi_1-\xi_2) = 0 \quad \text{------}(31).$$

27. Let

and

$$\left.\begin{aligned} m &\equiv m_1 + m_2, \\ \xi &\equiv \xi_1 - \xi_2. \end{aligned}\right\} \quad \text{------------} (32)$$

Then,

$$\begin{aligned} m_1 &= \frac{\mu m}{1+\mu}, \\ m_2 &= \frac{m}{1+\mu}, \\ \text{and} \quad m_1 m_2 &= \frac{\mu m^2}{(1+\mu)^2}. \end{aligned} \quad \text{............} (33)$$

For convenience, place

$$\Delta \equiv \frac{(1+\mu)^2}{\mu m}. \quad \text{............} (34)$$

Equation (31) may now be written as

$$\frac{d^2\xi}{dt^2} + \Delta b\frac{d\xi}{dt} + \Delta a\xi = 0. \quad \text{......} (35)$$

The solution of this equation is

$$\xi = C_1 \varepsilon^{-\frac{\Delta bt}{2} + \frac{1}{2}\sqrt{(\Delta b)^2 - 4a\Delta}\,t} + C_2 \varepsilon^{-\frac{\Delta bt}{2} - \frac{1}{2}\sqrt{(\Delta b)^2 - 4a\Delta}\,t}, \quad \text{---} (36)$$

where C_1 and C_2 are constants. If the motion is to be oscillatory we must have

$$4a\Delta > \Delta^2 b^2.$$

Let

$$q \equiv \frac{1}{2}\sqrt{4a\Delta - \Delta^2 b^2}.$$

Then, the appropriate solution of equation (35) may be written in the form

$$\xi = \varepsilon^{-\frac{\Delta bt}{2}}\left[G_1 \cos qt + G_2 \sin qt\right], \quad \text{---} (37)$$

where G_1 and G_2 are constants. By a well-known procedure equation (37) may be written in the more compact form:

$$\xi = G\varepsilon^{-\frac{\Delta b t}{2}} \cos(qt - \varphi), \quad \text{------} \quad (38)$$

where G and φ are constants. The ratio of two successive amplitudes is $\varepsilon^{\Delta b \tau/2}$, and the logarithmic decrements, κ, is given by

$$\kappa = \frac{\Delta b \tau}{2}, \quad \text{----------} \quad (39)$$

where τ is the period of free vibration of the oscillator. By substituting the value of τ, namely,

$$\tau = \frac{2\pi}{\sqrt{4a\Delta - \Delta^2 b^2}}, \quad \text{--------} \quad (40)$$

in (39), and the value of Δ given by (34) in the resulting expression as well as in the following expressions which we get for a and b :

$$4a = \frac{4\pi^2 + \tau^2\Delta^2 b^2}{\Delta\tau^2}, \quad \text{------} \quad (41)$$

$$b^2 = \frac{4\kappa^2}{\Delta^2\tau^2}, \quad \text{-------} \quad (42)$$

we find that the constants a and b are given by

$$a = \frac{\pi^2 + \kappa^2}{\Delta\tau^2}, \quad \text{---------} \quad (43)$$

$$b = \frac{2\kappa}{\Delta\tau}. \quad \text{.........} \quad (44)$$

28. If we now substitute the values of a and b in eq. (25) we find that $\omega^4 Q^2$ which occurs in eq. (28) is given

by

$$\omega^4 Q^2 = \frac{[X - Y]^2 + 4\kappa^2 \omega^6 \tau^6 \mu^2 (1-\mu)^4}{m^2 [\mu^2(\pi^2+\kappa^2)^2 + 4\mu^2\kappa^2\omega^2\tau^2 + \mu^2\omega^4\tau^4 - 2\mu^2\omega^2\tau^2(\pi^2+\kappa^2)]^2} \quad \text{-------} (45)$$

where

$$X \equiv \mu\tau^2\omega^2(\pi^2+\kappa^2)(1+\mu^2+6\mu),$$

and

$$Y \equiv 16\kappa^2\omega^2\tau^2\mu^2 + 4\mu^2(\pi^2+\kappa^2)^2 + \mu\omega^4\tau^4(1+\mu)^2.$$

Let us place

$$\delta \equiv \omega\tau = \omega/\nu_0, \quad \text{---------} (46)$$

where ν_0 is the natural frequency of the oscillator. Equation (45) may then be written, after some simplifications, as

$$\omega^4 Q^2 = \frac{\{M(\pi^2+\kappa^2) - S\}^2 + 4\kappa^2\delta^6(1-\mu)^4}{m^2\mu^2[(\pi^2+\kappa^2)^2 + \delta^4 + 2\delta^2(\kappa^2-\pi^2)]^2}, \quad (47)$$

where

$$M \equiv \delta^2(1+\mu^2+6\mu) - 4\mu(\pi^2+\kappa^2),$$

and

$$S \equiv 16\mu\kappa^2\delta^2 + \delta^4(1+\mu)^2.$$

29. If κ and δ are small, we find to a sufficient degree of approximation, that

$$\frac{\omega^4 Q^2}{16} = \frac{1}{m^2}(1 - g_1\delta^2 + g_2\delta^4), \quad (48).$$

where

$$\left.\begin{aligned} g_1 &\equiv \frac{1+\mu^2-2\mu}{2\mu\pi^2}, \\ g_2 &\equiv \frac{\mu^4-12\mu^3+22\mu^2-12\mu+1}{16\mu^2\pi^4} \\ \text{and} \quad & g_2\delta^4 << g_1\delta^2 < 1 \end{aligned}\right\} \text{-----} (49)$$

30. Thus, by vitue of eqs. (27) and (48), the ratio of the intensity of the scattered radiation at a point distant r from the center of the electron and making an angle θ with the incident beam, to that of the incident radiation is given by

$$v = \frac{e^4(1+\cos^2\theta)}{2m^2r^2c^4}(1-g_1\delta^2+g_2\delta^4), \text{----} (50)$$

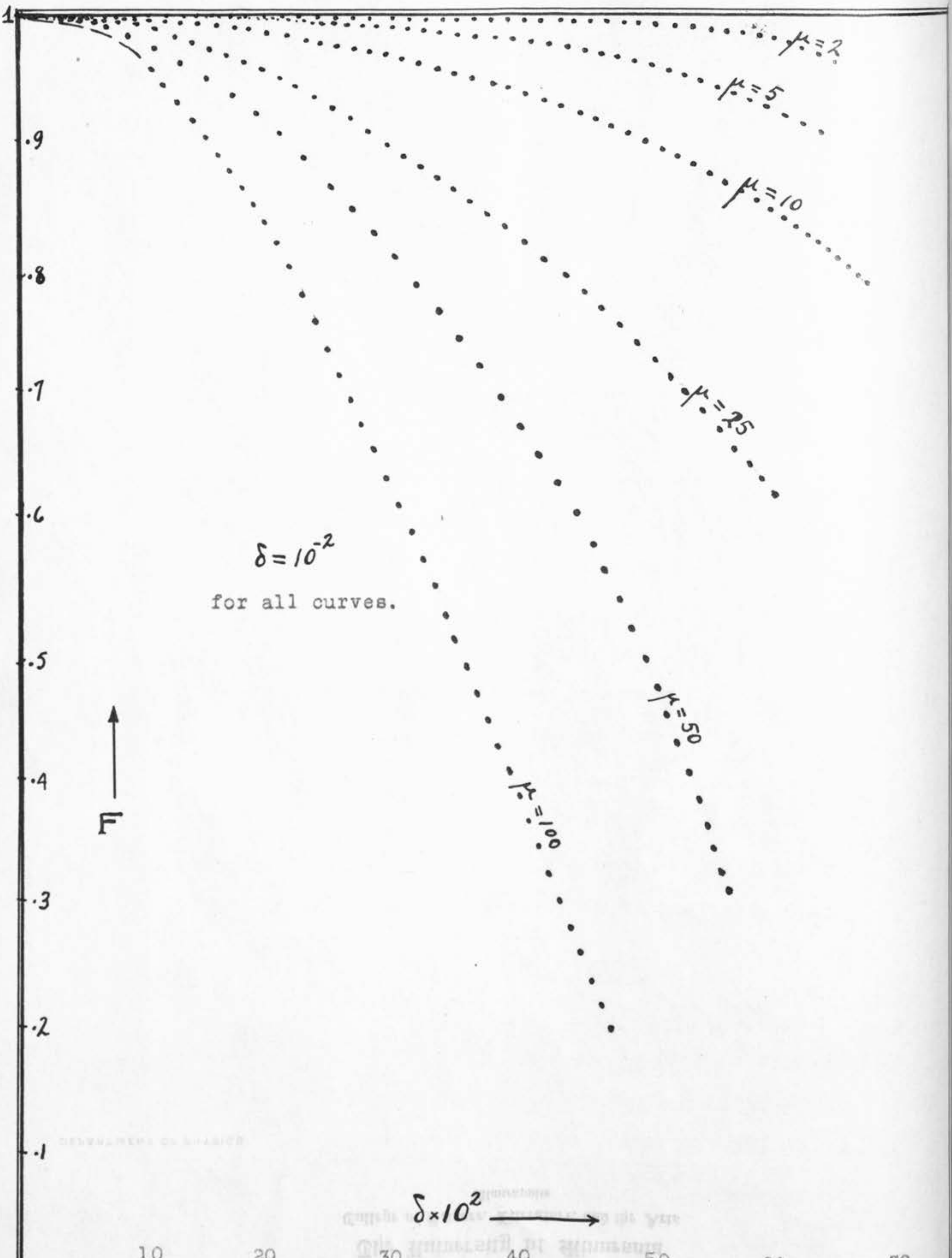
and the coefficient of scattering per electron, σ_1, is seen, by virtue of eqs. (28) and (48), to be given by

$$\sigma_1 = \frac{8\pi e^4}{3m^2c^4}(1-g_1\delta^2+g_2\delta^4). \text{....} (51)$$

If there are N atoms per cc. and p electrons per atom, the coefficient of scattering for the Np electrons is

$$\sigma = \frac{8\pi e^4 Np}{3m^2c^4}(1-g_1\delta^2+g_2\delta^4). \text{......} (52)$$

31. It is evident that the coefficient of scattering given by (52) reduces to the Thomosonian one when δ is negligibly small, that is, when the wave-length, λ,



1
.9
.8
.7
.6
.5
.4
.3
.2
.1
F
$\mu = 2$
$\mu = 5$
$\mu = 10$
$\mu = 25$
$\mu = 50$
$\mu = 100$
$\delta = 10^{-2}$
for all curves.
$\delta \times 10^2$
10
20
30
40
50
60
70

FIG. 6.

of the incident radiation is large. When, however, λ is not sufficiently large, δ is not negligible and according to (52) there should be a diminution in the scattering coefficient when short waves are used. This decrease in σ will depend on the values assumed for μ and δ, and by this two-fold choice of arbitrary constants one should be able to find a theoretical curve for the variation in σ with the wave-length which agrees, as well as may be, with the experimental results. If we divide both sides of eq. (52) by

$$\sigma_c = \frac{8\pi e^4 N p}{3m^2c^4},$$

which is the value of σ according to the classical theory, we obtain the quantity

$$F \equiv (1 - g_1\delta^2 + g_2\delta^4), \quad \text{------} (53)$$

which is the factor by which Thomson's expression for σ is to be multiplied in order to find the observed scattering coefficient. A series of curves for F, using different values of μ, are given in FIG.6.

33. It will be observed that while the present theory gives a scattering coefficient depending on the wave-length of the incident radiation, it predicts a distribution of the intensity of the scattered beam which is symmetrical with respect to the radiating plate. This, as we have already remarked, is not borne out by

experiment.

33. We have seen that the observed decrease in the total scattering may be explained in terms of the properties of the electron itself. In the endeavor to explain the observed asymmetry also in terms of the properties of the electron, the writer was led to consider various models of the electron and certain cumulative effects. Among these may be mentioned (1)the "shielding effect"(i.e., the fact that the force acting on any electron will not merely be that due to the incident beam, but, in addition, there will be a contribution from all other electrons in the neighborhood of the one considered, due to their motion)";(2)the effect of the magnetic force of the incident beam; (3)an electron consisting of a doublet and which radiates only in a certain direction; and (4)an electron consisting of two parts having a doublet superimposed upon one of them. Investigation has shown that the fore and aft asymmetry cannot consistently be explained on the basis of any of these effects.

34. While there may be more than one way of explaining the diminution in σ when sufficiently short waves are used, it does not seem possible to adhere with any degree of closeness to classical electrodynamics and explain both the decrease in σ and the fore and aft asymmetry.

35. It was next thought desirable to combine the theory outlined above with that of Debye. This procedure is legitimate because the compound oscillator gives a field which is the same function of θ as that due to a Thomsonian electron. The changes which are necessary to make in equations (3) to (13) are obvious if we take care to define the vector U such that[10] its z-component is

$$U_z \equiv U_{z_1} + U_{z_2} = \left(\frac{e}{2r}\xi_{m_1}\varepsilon^{-ikr}\right) + \left(\frac{e}{2r}\xi_{m_2}\varepsilon^{-ikr}\right).$$

Instead of considering the displacement (ξ_m) of the <u>nth</u> electron as a whole, we will consider this displacement to be componded, vectorially, of the displacements of the separate parts of this electron. It is then merely necessary to replace (e/mc^2) in equations (3) to (7) by

$$k^2\sqrt{(4e^2/m^2\omega^4)(1-g_1\delta^2+g_2\delta^4)},$$

where, as above, $k = \omega/c$. The result of introducing the indicated change is that the ratio of the intensity of the scattered beam to that of the incident beam is given by

$$V = \frac{Npe^2g^2k^4}{4R^2}\left(\frac{1+\cos^2\theta}{2}\right)\sum_{n=0}^{p-1}\frac{\sin[4ka\sin\frac{n\pi}{p}\sin\frac{\theta}{2}]}{[4ka\sin\frac{n\pi}{p}\sin\frac{\theta}{2}]} \qquad (54)$$

where

$$g^2/4 \equiv e^2(1-g_1\delta^2+g_2\delta^4)/m^2\omega^4,$$

and the other symbols have the same significance as before.

36. The scattering coefficient may be obtained by integration and comes out to be

$$\sigma = \frac{Np e^4 (1-g_1\delta^2+g_2\delta^4)}{m^2c^4}\int VR^2 d\Omega \ . \quad \text{.... (56)}$$

This expression leads to two special cases corresponding to those already considered above.

Case 1: If the wave-length, λ, of the incident beam is much greater than the radius of the ring(a), then

$$\sigma = \frac{8\pi e^4}{3m^2c^4}(1-g_1\delta^2+g_2\delta^4)Np^2 \quad \text{...... (57)}$$

Case 2: When $\lambda << a$, then

$$\sigma = \frac{8\pi e^4}{3m^2c^4}(1-g_1\delta^2+g_2\delta^4)Np \ . \quad \text{...... (58)}$$

It will be observed that equations (54) to (58) predict "excess scattering", fore and aft asymmetry, and a decrease in the scattering coefficient.

37. The function

$$\Phi_{p,\delta}(j) = (1-g_1\delta^2+g_2\delta^4)\frac{1}{p}\sum_{n=0}^{p-1}\frac{\sin[j\sin\frac{n\pi}{p}]}{[j\sin\frac{n\pi}{p}]}, \quad (59)$$

where $j = 4ka \sin\frac{\theta}{2}$ is displayed graphically in FIG.3 for different values of p, j, and δ. In FIG.7 are given the theoretical intensity-distribution curves according to the classical theory and also according to the modified Debye theory for different values of δ, and 4ka. FIG.5 gives the experimental distribution

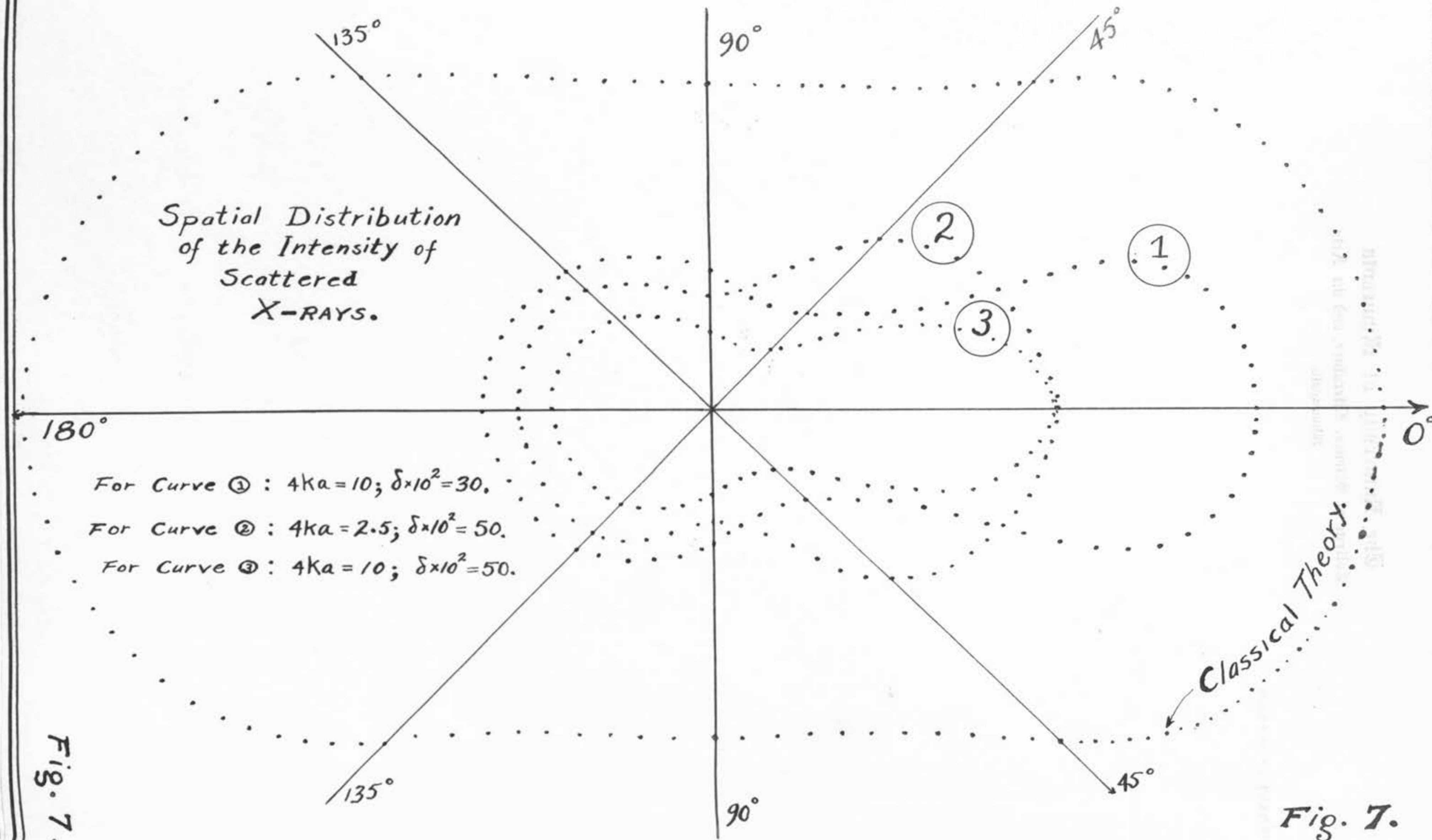

Spatial Distribution of the Intensity of Scattered X-RAYS.
135°
90°
45°
180°
0°
1
2
3
For Curve ①: 4ka = 10; δ×10² = 30.
For Curve ②: 4ka = 2.5; δ×10² = 50.
For Curve ③: 4ka = 10; δ×10² = 50.
Classical Theory
135°
90°
45°
Fig. 7

Fig. 7.

curve for Al as found by Crowther[14] and the theoretical distribution curves for Hydrogen according to the modified and unmodified Debye's theory. It will be seen that, although the curves are plotted for different substances, the experimental and theoretical curves are, at any rate as regards their form, in excellent agreement.

38. It is desirable to record here that instead of assuming that the electron consists of two parts each having a charge of $(-e/2)$, we might have assumed that the two parts were unequally charged; e.g., one part could have a charge of $-2e$ and the other part a charge of $+e$. Such an assumption has, in fact, been made, and it will be of interest to state briefly the differences between this oscillator and the one already discussed.

39. Using the same notation as before, the equations of motion of this oscillator are assumed to be:

$$\left.\begin{aligned} m_1\ddot{\xi}_1 + b(\dot{\xi}_1-\dot{\xi}_2) + a(\xi_1-\xi_2) &= -2eE_0\,\varepsilon^{i(\omega t - kx)} \\ m_2\ddot{\xi}_2 + b(\dot{\xi}_2-\dot{\xi}_1) + a(\xi_2-\xi_1) &= +eE_0\,\varepsilon^{i(\omega t - kx)} \end{aligned}\right\} \quad (60)$$

Solving these equations for $\ddot{\xi}_1$ and $\ddot{\xi}_2$ and combining them as we have done above, we find that the intensity of radiation is given by

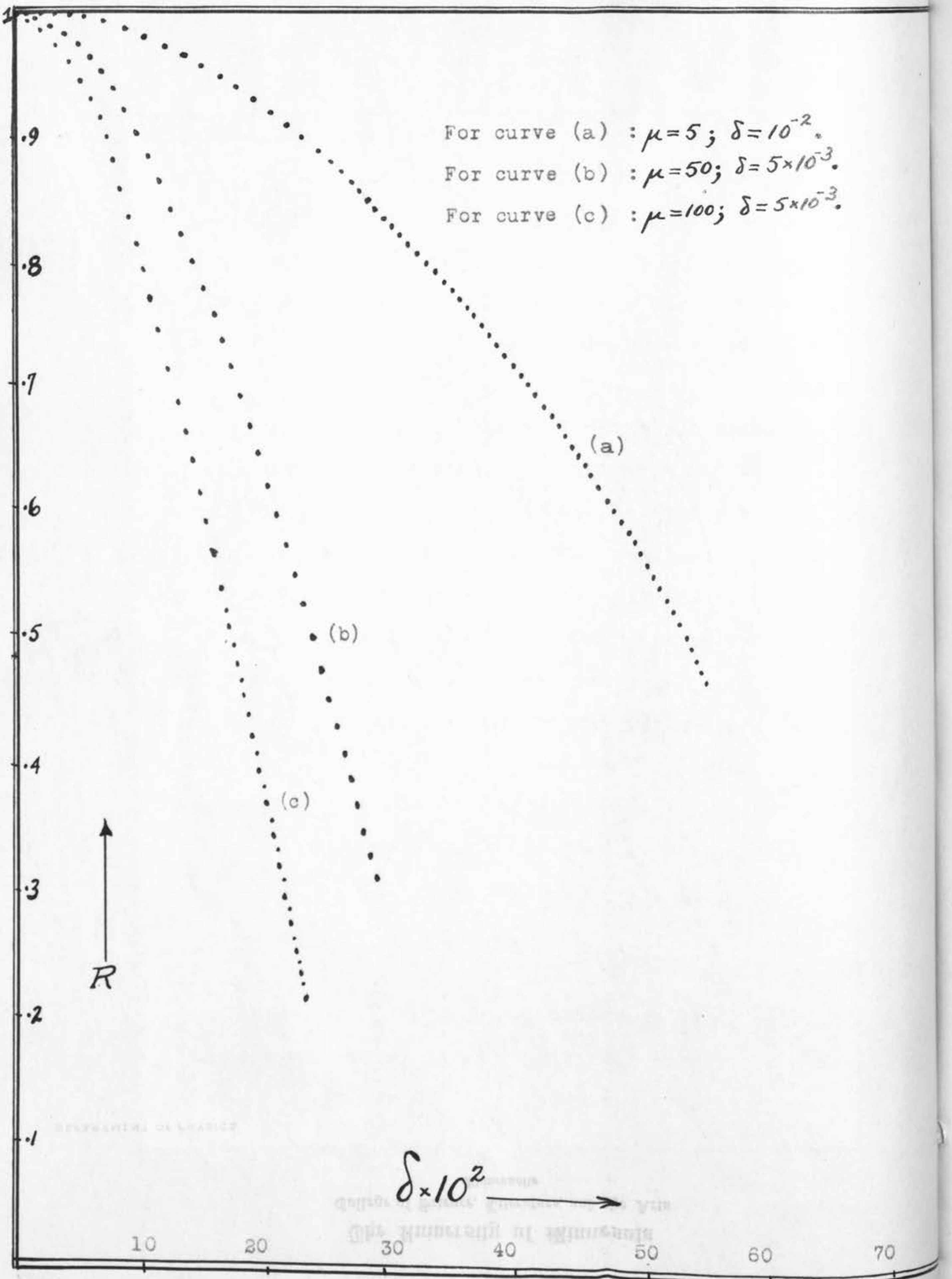

For curve (a) : $\mu=5$; $\delta=10^{-2}$.
For curve (b) : $\mu=50$; $\delta=5\times10^{-3}$.
For curve (c) : $\mu=100$; $\delta=5\times10^{-3}$.
(a)
(b)
(c)
R
$\delta\times10^2$
1
.9
.8
.7
.6
.5
.4
.3
.2
.1
10
20
30
40
50
60
70

FIG. 8.

$$I_\theta = \frac{e^4(1+\cos^2\theta)}{2r^2c^4}\,\omega^4 E_0^2 P^2, \quad \text{------} \quad (61)$$

where

$$P^2 = \frac{\left\{(1+\mu)\left(\frac{b^2}{m_2}+\frac{a^2}{m_2\omega^2}\right) - a(\mu^2+6\mu+4) + m_1\omega^2(\mu+4)\right\}^2 + b^2\omega^2(\mu+2)^4}{\left[a^2(1+\mu)^2 + m_1\omega^4 - 2a(1+\mu)m_1\omega^2 + b^2\omega^2(1+\mu)^2\right]^2} \quad \text{------} \quad (62)$$

Introducing, as before, the logarithmic decrement, κ , and the natural frequency, ν_0 , of this compound oscillator, we find, for sufficiently small values of κ and δ,

$$\omega^4 P^2 = \frac{1}{m^2}\left[1 - \left(\frac{2\mu^2+8\mu+8}{\mu\pi^2}\right)\delta^2 + \left(\frac{\mu^4+6\mu^3-8\mu^2+24\mu+16}{\mu^2\pi^4}\right)\delta^4\right],$$

where, as above, $\delta \equiv \omega/\nu_0$.

40. The scattering coefficient per electron is given by

$$\sigma_1 = \frac{8\pi e^4}{3m^2c^4}\left(1 - h_1\delta^2 + h_2\delta^4\right), \quad \text{----} \quad (63)$$

where

$$\left.\begin{aligned} h_1 &\equiv \frac{2\mu^2+8\mu+8}{\mu\pi^2}, \\ \text{and} \quad h_2 &\equiv \frac{\mu^4+6\mu^3-8\mu^2+24\mu+16}{\mu^2\pi^4}. \end{aligned}\right\} \quad \text{------} \quad (64)$$

If we divide this expression by the value of σ given by the classical theory we obtain the factor

$$R \equiv \left(1 - h_1\delta^2 + h_2\delta^4\right). \quad \text{-------} \quad (65)$$

The values of R are plotted against δ for different values of μ and δ in FIG.8. Here again it is seen that the diminution in the scattering coefficient may be accounted for; but, as in the previous case, a theory based on a scattering unit of this kind cannot explain the dissymmetry of the spatial distribution of the intensity of the scattered radiation.

41. Upon inspection of equations (60) to (63) and (19) to (58) it will be evident that the combination of the present modified electron with Debye's theory would give similar results to those resulting from the combination already discussed. There is one difference between

THE CONSERVATION OF ENERGY.

42. We shall now consier briefly the difficulties concerning the energy equation which were mentioned in Paragraph 23. The term $b(\dot{\xi}_1 - \dot{\xi}_2)$ is akin to a frictional term and gives rise to a dissipation of energy. Except for small accelerations the compound oscillators considered above do not obey the conservation of energy in the strict sense of the word. However, this difficulty is not peculiar to these oscillators alone. It is well-known[16], for example, that the simple Lorentzian electron does not obey the energy equation*. The question

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*See, however, a discussion by W.F.G.Swann in the Bull. of the Nat. Research Council, No. 24, 1922.

of whether the energy equation is obeyed or not is a question of what we assume for the force equation. Any departure from the usual force equation may be expected to give rise to difficulties concerning the energy equation. We are not, however, forced to the conclusion that the conservation of energy is not true. The failure of the force equation assumed above to comply with the requirements of the equation of energy, simply means that the quantity we have usualy called "energy" is not the one which is conserved for the oscillators discussed in the present paper. It would be an interesting problem for future investigation to see just what quantities must be called "kinetic energy" and "work" in order that the equation of energy shall be valid for all accelerations. For the present, we shall content ourselves with merely recording the order of magnitude of the energy dissipated. For the oscillator consisting of two equally charged components of an electron, a simple calculation has shown that if the constant b in equation (14) and (15) is of the order 10^{-3}, and μ of the order 10, the energy per sq.cm. of the incident beam which is dissipated is of the order of .5 per cent. As far as any experimental evidence is concerned there may actually be a dissipation of energy in the atom, and if so, it may take place in some such fashion as that indicated in this paper.

43. It will be instructive, following Schott, to compare, in tabulated form, the various theories proposed for explaining the scattering of X-rays.

Theory.	Max. total scattering (long λ's) is proportional to	Min. total scattering (short λ's) is proportional to	Max. asymmetry.
Thomson's	Np	Np	0
Electron ring	$N\sum p_i^2$	Np	Increases with p
Ring electron	Np	0	Always large.
Present work combined with electron ring	$N\sum p_i^2$	0	Increases somewhat with p

In the above, N is the number of atoms per cc., and p_i is the number of electrons in the <u>ith</u> ring, and $\sum p_i^2$ is the summation of p_i^2 taken over all the rings in the atom if the latter consists of more than one ring.

44. To review, we may say that in the scattering of X-rays of short wave-lengths by the lighter elements the observed scattering coefficient is much less than that

predicted by the classical theory. It has thus far not been possible to explain satisfactorily this diminution in the scattering coefficient on the basis of classical electrodynamics. The work herein presented shows how, by a slight modification of the ordinary equation of motion of an electron, we may account for the observed decrease in the scattering coefficient; and that by considering each electron as a sort of compound oscillator and making use of Debye's theory we may also account for the observed asymmetrical distribution of the intensity of the scattered beam. In this investigation, the quest was not for perfect agreement between theoretical predictions and the experimental results on one or two elements, as our experimental knowledge of the scattering of X-rays does not at all warrant such a procedure. The aim has rather been to discuss the possibility of modifying the classical theory so that it may lead to a reasonable explanation of the observed facts.

The writer deems it a pleasure to acknowledge his indebtedness to Prof.W.F.G. Swann, for suggesting the problem, for criticisms, suggestions, and encouragement throughout the progress of the work.

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